MCT 211
Automatic Control

Frequency Response Analysis

Faculty of Engineering

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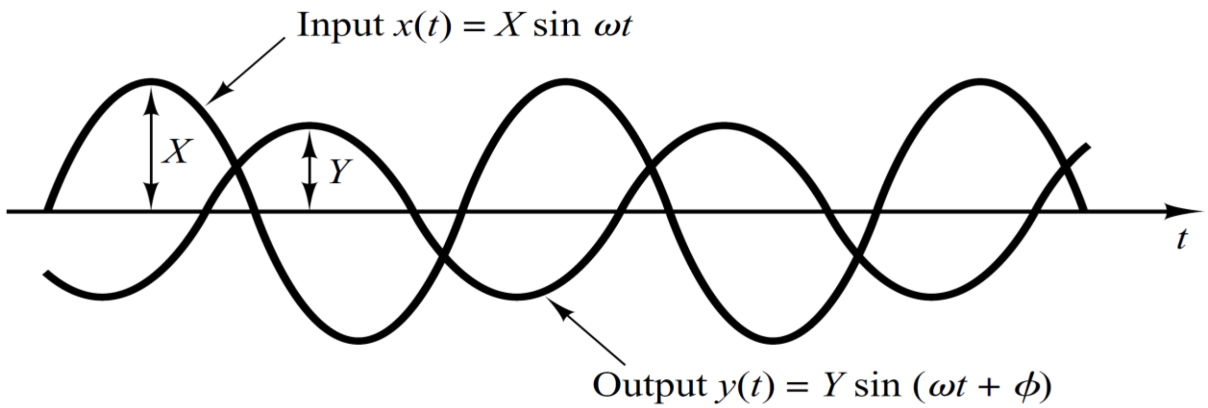
Frequency Response Analysis

Introduction

- The **frequency response** of a system is defined as the **steady-state response** of the system to a **sinusoidal input** signal.
- The sinusoid is a unique input signal, and the resulting **output** signal for a linear system is **sinusoidal** in the **steady state**; it **differs** from the input only in **amplitude** and **phase** angle.



Frequency Response Analysis

Introduction

The frequency response can be calculated by replacing ***s*** in the transfer function by ***Jω***.

$$G(j\omega) = Me^{j\phi} = M\,\underline{/\phi}$$

M is the **amplitude ratio** of the output and input sinusoids and ***ϕ*** is the **phase shift** between the input sinusoid and the output sinusoid

$$|G(j\omega)| = \left|\frac{Y(j\omega)}{X(j\omega)}\right| \qquad \underline{/G(j\omega)} = \underline{/\frac{Y(j\omega)}{X(j\omega)}}$$

Frequency Response Analysis

Example

$$\phi = \underline{/G(j\omega)} = \tan^{-1}\left[\frac{\text{imaginary part of } G(j\omega)}{\text{real part of } G(j\omega)}\right]$$

$$x(t) = X\sin\omega t \longrightarrow G(s) = \frac{K}{Ts+1} \longrightarrow G(j\omega) = \frac{K}{jT\omega + 1}$$

The amplitude ratio is $\longrightarrow$

$$|G(j\omega)| = \frac{K}{\sqrt{1+T^2\omega^2}}$$

The phase shift is $\longrightarrow$

$$\phi = \underline{/G(j\omega)} = -\tan^{-1} T\omega$$

$$y_{ss}(t) = \frac{XK}{\sqrt{1+T^2\omega^2}}\sin(\omega t - \tan^{-1}T\omega)$$

Frequency Response Analysis

Presenting Frequency-Response

1. Bode diagram or logarithmic plot
2. Nyquist plot or polar plot
3. Log-magnitude-versus-phase plot (Nichols plots)

Frequency Response Analysis

Bode Plot

A Bode diagram consists of two graphs:

- **logarithm of the magnitude** of a sinusoidal transfer function; **the phase angle.**
- both are plotted against the **frequency** on a **logarithmic** scale.
- Logarithmic magnitude of ***G(jω)*** is ***20 log |G(jω)|***, where the base of the logarithm is **10**.
- The **unit** used in the magnitude is the decibel, usually abbreviated ***dB***.
- Drawn on **semi-log** paper, using the **log scale** for **frequency** and the **linear** scale for either **magnitude** (in decibels) or **phase** angle (in degrees).

Frequency Response Analysis

Bode Plot

Gain K

$$20 \log K_b = \text{constant in dB} \longrightarrow \phi(\omega) = 0$$

The gain curve is a horizontal line at the magnitude of $20\log K_b$ decibels.

If the gain is a negative value, $-K_b$, the logarithmic gain remains $20 \log K_b$. The negative sign is accounted for by the phase angle, $-180°$.

The effect of varying the gain K_b in the transfer function is that it raises or lowers the log-magnitude curve of the transfer function by the corresponding constant amount, but it has no effect on the phase curve.

Frequency Response Analysis

Bode Plot

Integral and Derivative Factors $(j\omega)^{\pm}$

Poles (or Zeros) at the Origin $(j\omega)$

$$20\log\left|\frac{1}{j\omega}\right| = -20\log\omega\ \text{dB} \longrightarrow \phi(\omega) = -90^\circ$$

The slope of the magnitude curve is −20 dB/decade for a pole.
Similarly, for a multiple pole at the origin, we have

$$20\log\left|\frac{1}{(j\omega)^N}\right| = -20N\log\omega \longrightarrow \phi(\omega) = -90^\circ N$$

The slope due to the multiple pole is $-20N$ dB/decade

Frequency Response Analysis

Bode Plot

Integral and Derivative Factors $(j\omega)^{\pm}$

Poles (or Zeros) at the Origin $(j\omega)$

$$20 \log|j\omega| = +20 \log \omega \longrightarrow \phi(\omega) = +90^\circ$$

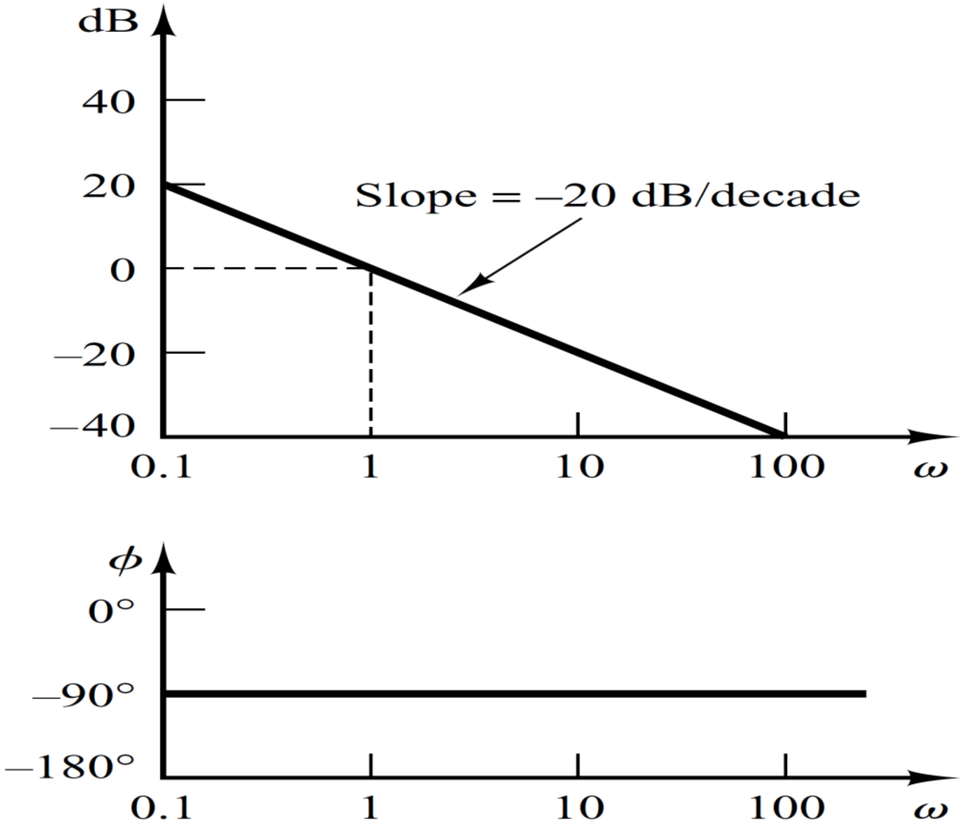
The slope of the magnitude curve is +20 dB/decade for a zero.

Frequency Response Analysis

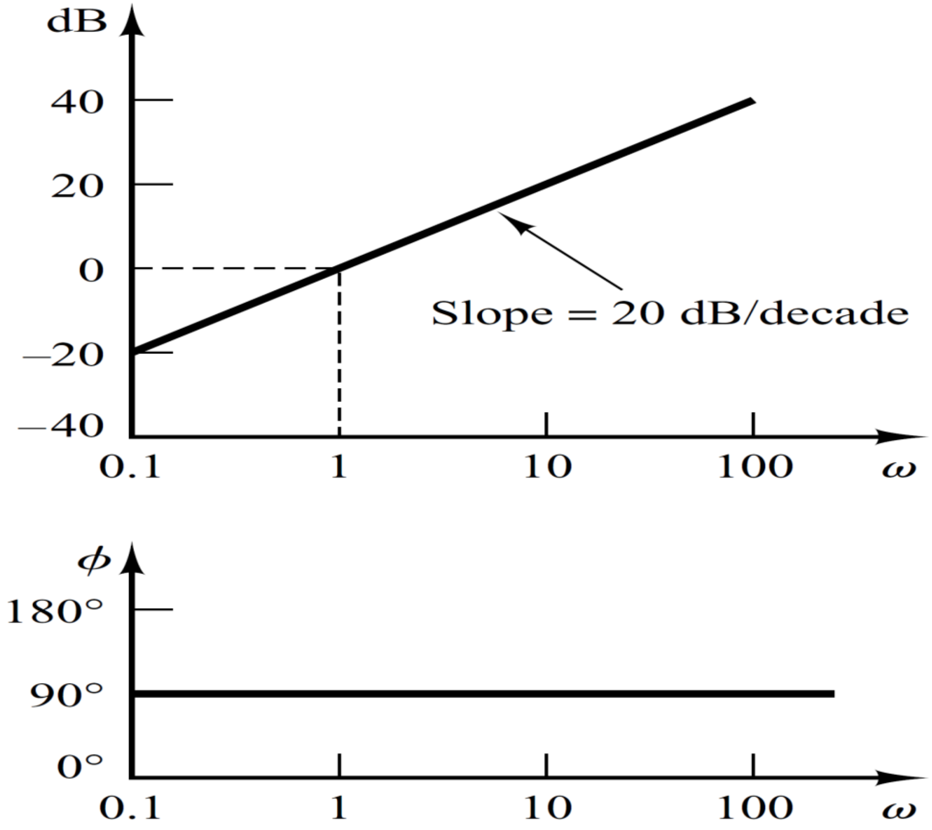
Bode Plot

Integral and Derivative Factors $(j\omega)^{\pm}$

Poles (or Zeros) at the Origin $(j\omega)$



Bode diagram of
$G(j\omega) = 1/j\omega$



Bode diagram of
$G(j\omega) = j\omega$

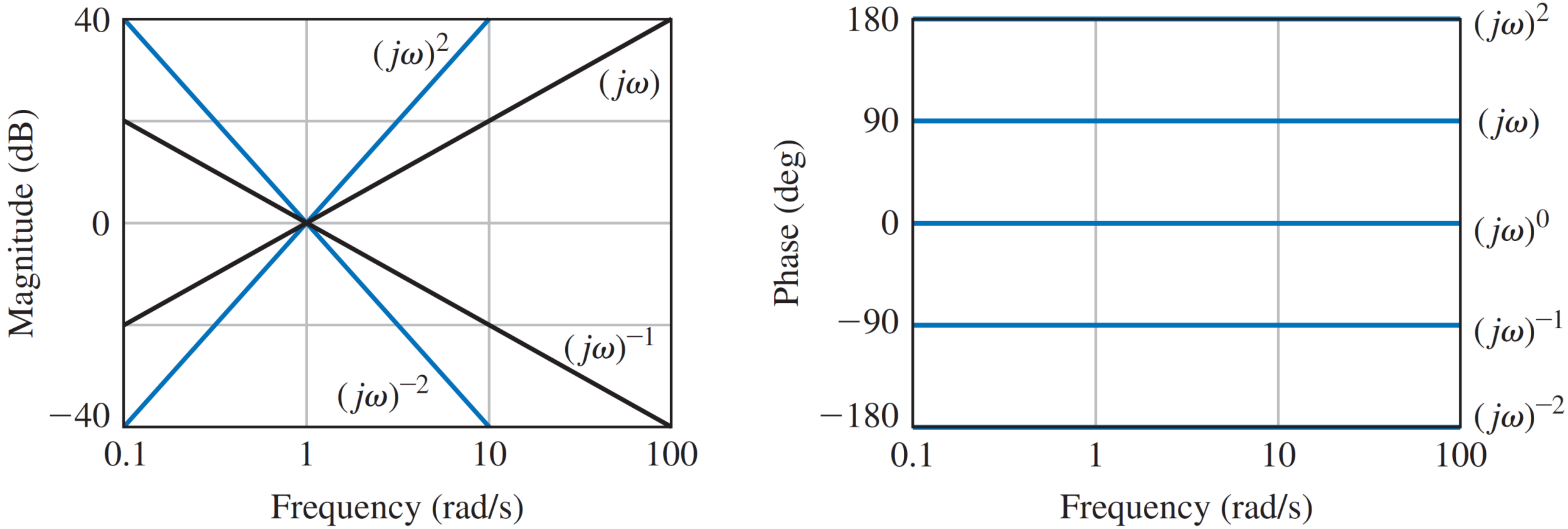
Frequency Response Analysis

Bode Plot

Integral and Derivative Factors $(j\omega)^{\pm}$

Poles (or Zeros) at the Origin $(j\omega)$

$$N = 1 \text{ and } N = 2$$



Frequency Response Analysis

Bode Plot

First-Order Factors $(1 + j\omega)^{\pm}$

Poles or Zeros on the Real Axis

$$20 \log \left| \frac{1}{1 + j\omega T} \right| = -20 \log \sqrt{1 + \omega^2 T^2} \text{ dB}$$

For low frequencies → $\omega \ll 1/T$ → $-20 \log \sqrt{1 + \omega^2 T^2} \doteqdot -20 \log 1 = 0 \text{ dB}$

For high frequencies → $\omega \gg 1/T$ → $-20 \log \sqrt{1 + \omega^2 T^2} \doteqdot -20 \log \omega T \text{ dB}$

At **ω=1/T**, the log magnitude equals **0 dB**; at **ω =10/T**, the log magnitude is **–20 dB**.
Thus, the value of **–20 log ωT dB** decreases by **20 dB** for every decade of **ω**.
For **ω >>1/T**, the log-magnitude curve is thus a straight line with a slope of **–20 dB/decade** (or **–6 dB/octave).**

Frequency Response Analysis

Bode Plot

First-Order Factors $(1 + j\omega)^{\pm}$

Poles or Zeros on the Real Axis

Decade: ω=10
Octave: ω=2

The logarithmic representation of the frequency-response curve of the factor $1/(1+j\omega T)$ can be approximated by two straight-line **asymptotes**, one a straight line at 0 dB for the frequency range $0<\omega<1/T$ and the other a straight line with slope –20 dB/decade (or –6 dB/octave) for the frequency range $1/T< \omega <\infty$.

The frequency at which the two asymptotes meet is called the ***corner frequency*** or ***break frequency***.

$$\omega = 1/T \longrightarrow -20\log\sqrt{1+1} + 20\log 1 = -10\log 2 = -3.03\ \text{dB}$$

Frequency Response Analysis

Bode Plot

First-Order Factors $(1 + j\omega)^{\pm}$

Poles or Zeros on the Real Axis

$$\phi = -\tan^{-1}\omega T$$ → ***At ω=0 → ϕ=0*** → ***At ω=∞ → ϕ= -90***

At ω= 1/T → $$\phi = -\tan^{-1}\frac{T}{T} = -\tan^{-1}1 = -45°$$

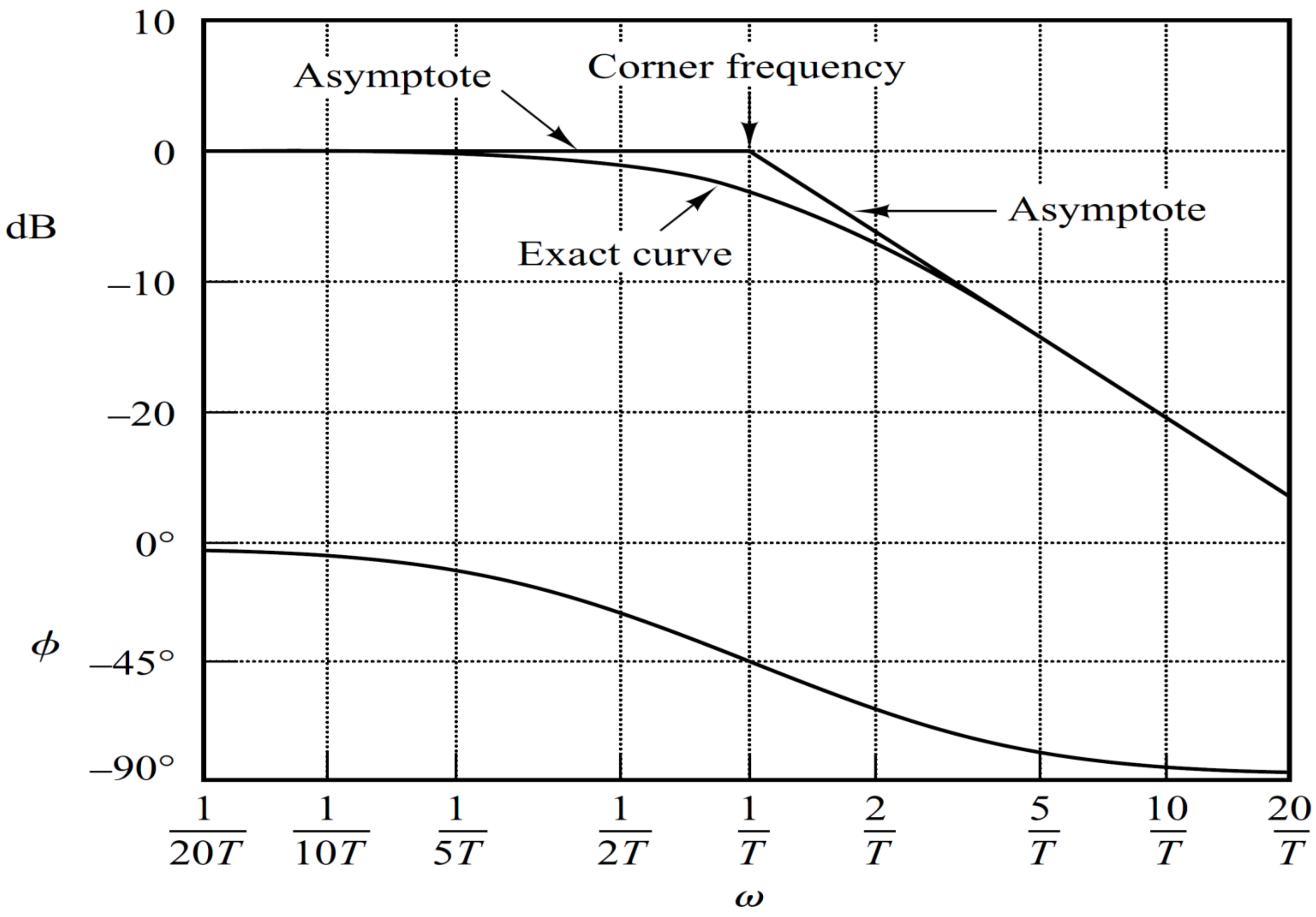
The phase angle is skew **symmetric** about the inflection point at ϕ =–45°.

Frequency Response Analysis

Bode Plot

First-Order Factors $(1 + j\omega)^{\pm}$

Poles or Zeros on the Real Axis



$$20\log\left|\frac{1}{1+j\omega T}\right| = -20\log\sqrt{1+\omega^2T^2}\ \text{dB}$$

$$\phi = -\tan^{-1}\omega T$$

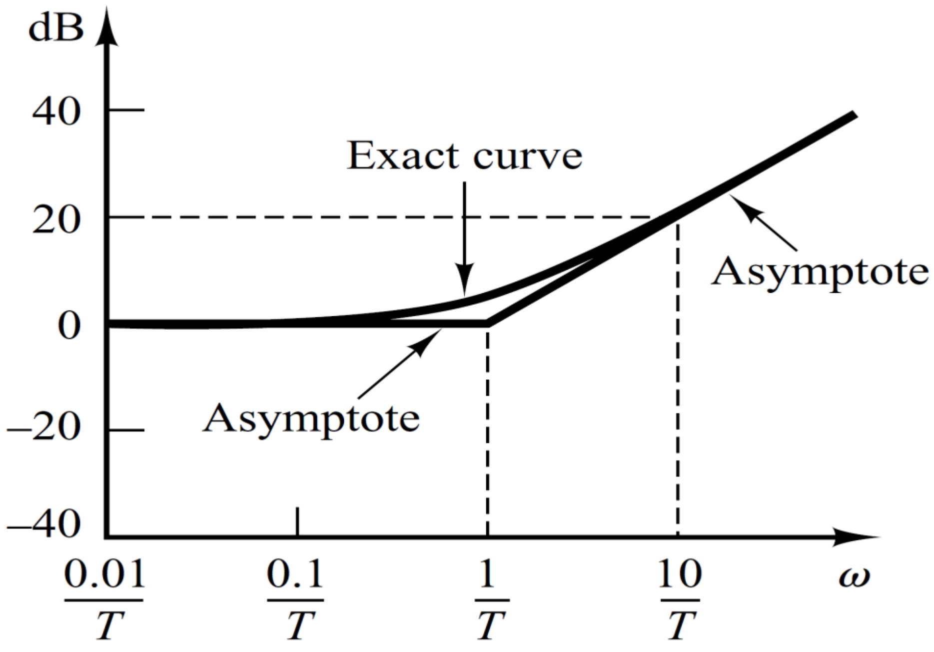
Frequency Response Analysis

Bode Plot

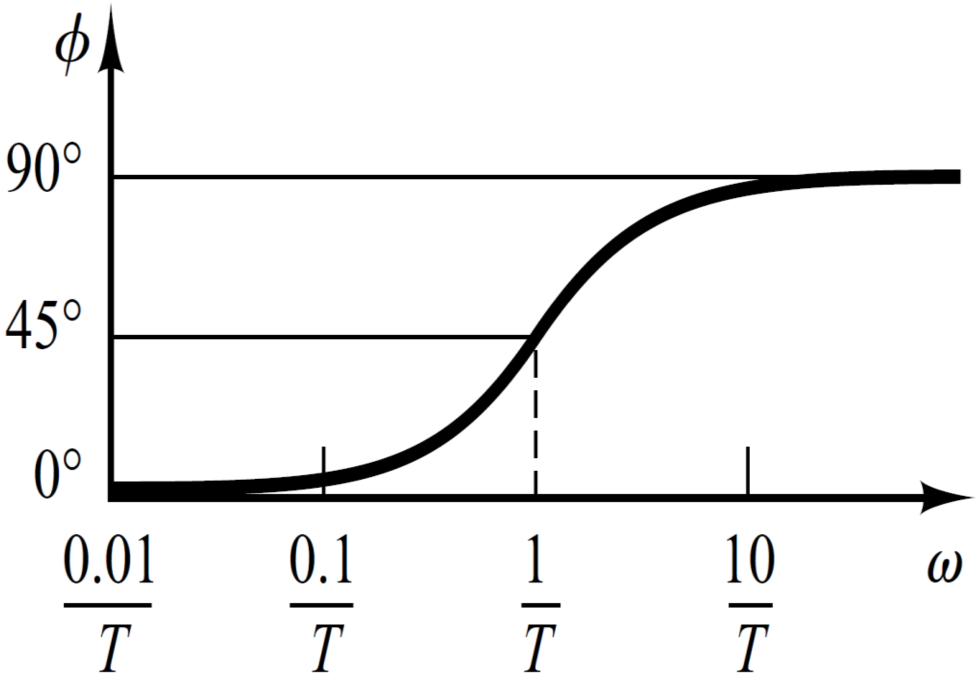
First-Order Factors $(1 + j\omega)^{\pm}$

Poles or Zeros on the Real Axis

$$20 \log|1 + j\omega T|$$



$$\underline{/1 + j\omega T} = \tan^{-1}\omega T$$



Frequency Response Analysis

Bode Plot

Quadratic Factors $[1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2]^{\pm}$

Complex Conjugate Poles or Zeros

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

- If $\zeta>1$, this quadratic factor can be expressed as a product of two first-order factors with real poles.
- If $0<\zeta<1$, this quadratic factor is the product of two complex conjugate factors.

$$20\log\left|\frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}\right| = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

Frequency Response Analysis

Bode Plot

Quadratic Factors $[1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2]^{\pm}$

Complex Conjugate Poles or Zeros

$$20\log\left|\frac{1}{1+2\zeta\left(j\frac{\omega}{\omega_n}\right)+\left(j\frac{\omega}{\omega_n}\right)^2}\right| = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

For low frequencies → $\omega \ll \omega_n$ → $-20\log 1 = 0\ \text{dB}$

For high frequencies → $\omega \gg \omega_n$ → $-20\log\frac{\omega^2}{\omega_n^2} = -40\log\frac{\omega}{\omega_n}\ \text{dB}$

Asymptote with slope -40 dB/decade

Frequency Response Analysis

Bode Plot

Quadratic Factors $[1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2]^{\pm}$

Complex Conjugate Poles or Zeros

$$\phi = \angle \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} = -\tan^{-1}\left[\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

At $\omega=0 \rightarrow \phi=0$

At $\omega=\infty \rightarrow \phi= -180$

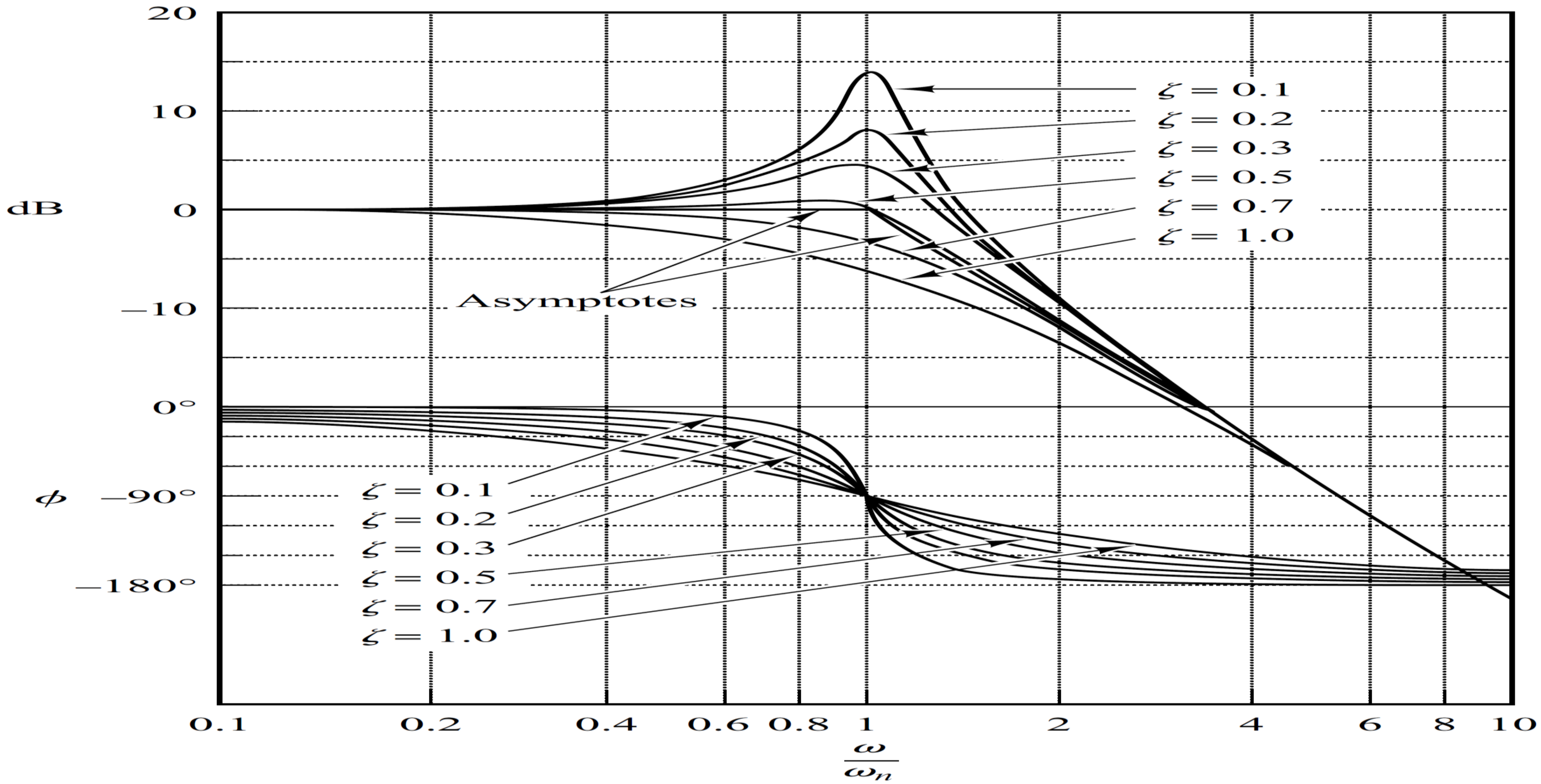
At $\omega=\omega_n$ $\longrightarrow$ $\phi = -\tan^{-1}\left(\frac{2\zeta}{0}\right) = -\tan^{-1}\infty = -90^\circ$

Frequency Response Analysis

Bode Plot

Quadratic Factors $[1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2]^{\pm}$

Complex Conjugate Poles or Zeros



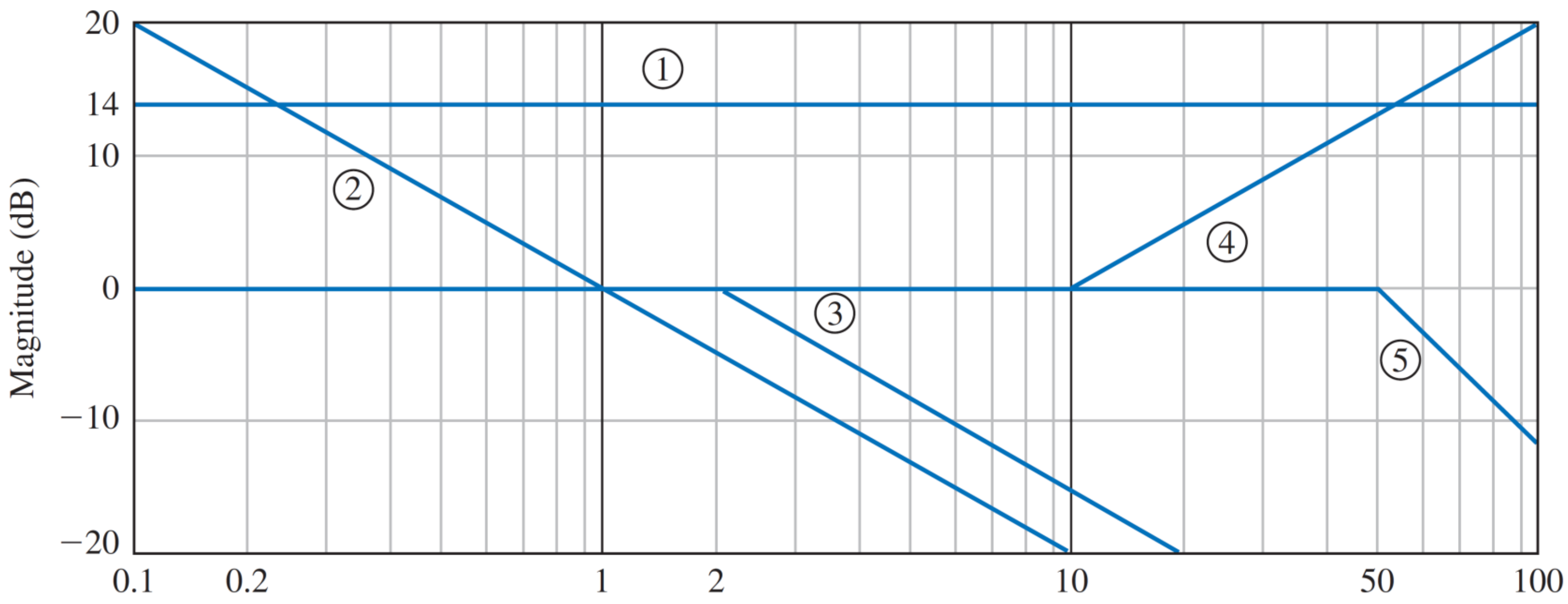
Frequency Response Analysis

Bode Plot

Example

$$G(j\omega) = \frac{5(1 + j0.1\omega)}{j\omega(1 + j0.5\omega)(1 + j0.6(\omega/50) + (j\omega/50)^2)}$$

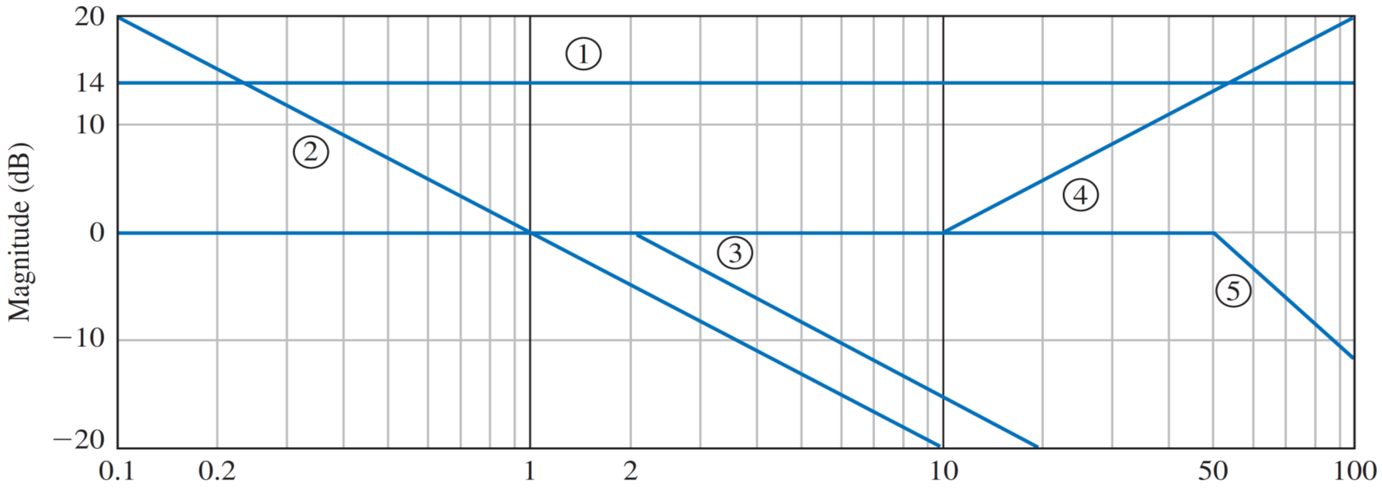
1. A constant gain $K = 5$
2. A pole at the origin
3. A pole at $\omega = 2$
4. A zero at $\omega = 10$
5. A pair of complex poles at $\omega = \omega_n = 50$.

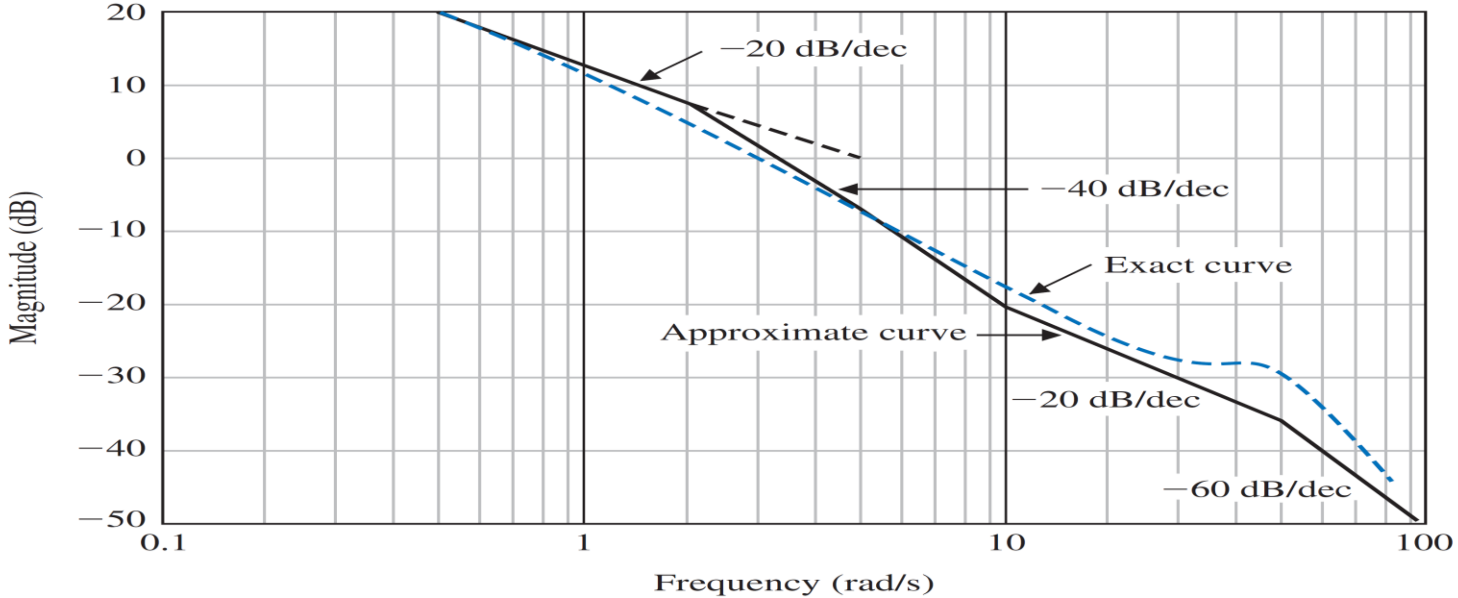


Frequency Response Analysis

Bode Plot

Example





1. A constant gain $K = 5$
2. A pole at the origin
3. A pole at $\omega = 2$
4. A zero at $\omega = 10$
5. A pair of complex poles at $\omega = \omega_n = 50$.

Frequency Response Analysis

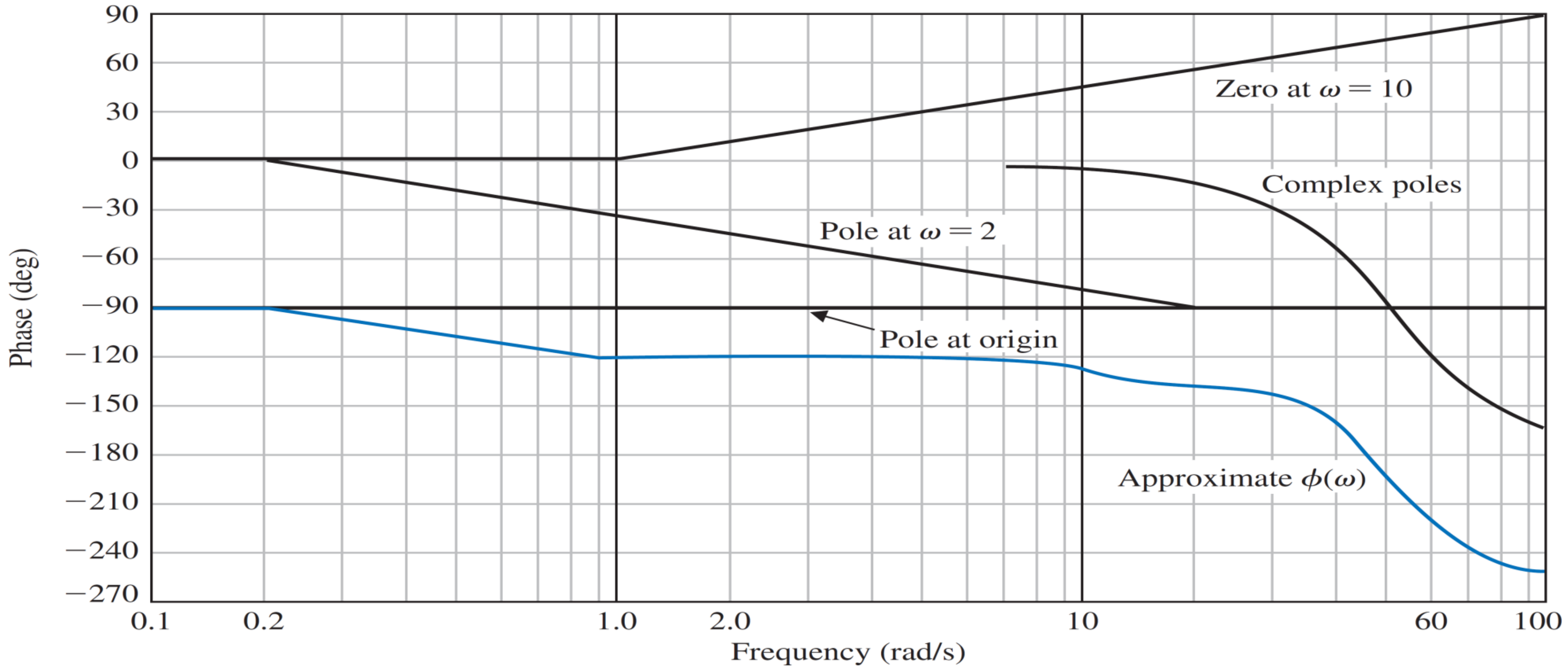
Bode Plot

Example

$$\phi(\omega) = -90^\circ - \tan^{-1}\omega\tau_1 + \tan^{-1}\omega\tau_2 - \tan^{-1}\frac{2\zeta u}{1-u^2}$$

$\tau_1 = 0.5, \quad \tau_2 = 0.1, \quad 2\zeta = 0.6,$ and $u = \omega/\omega_n = \omega/50.$

1. A constant gain $K = 5$
2. A pole at the origin
3. A pole at $\omega = 2$
4. A zero at $\omega = 10$
5. A pair of complex poles at $\omega = \omega_n = 50.$



Frequency Response Analysis

Bode Plot

Minimum-Phase Systems and Nonminimum-Phase Systems

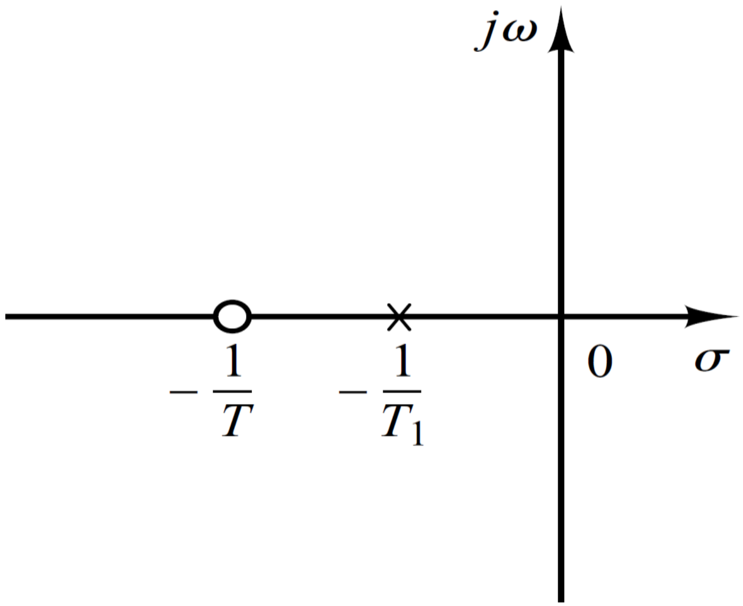
- **A transfer function is called a minimum phase transfer function if all its zeros lie in the left-hand *s*-plane.**
- **It is called a nonminimum phase transfer function if it has zeros in the right-hand *s*-plane.**
- A minimum-phase system, the magnitude and phase-angle characteristics are uniquely related.
- This, however, does not hold for a nonminimum-phase system.

Frequency Response Analysis

Bode Plot

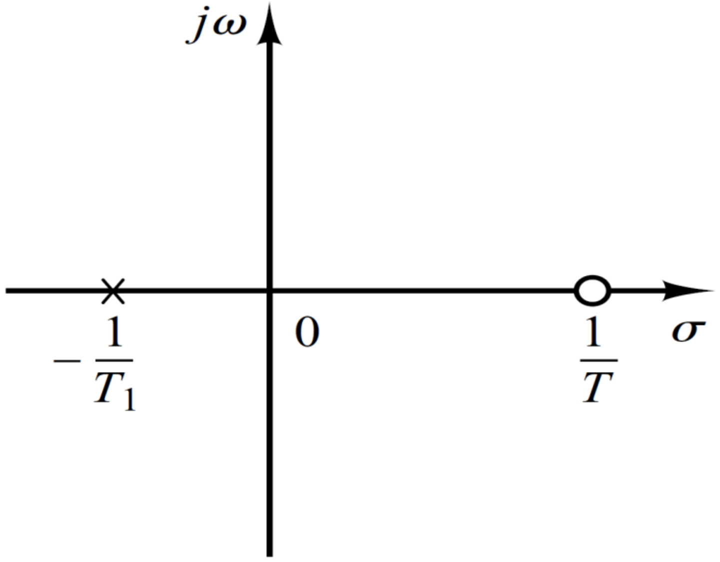
Minimum-Phase Systems and Nonminimum-Phase Systems

$$G_1(j\omega) = \frac{1 + j\omega T}{1 + j\omega T_1}$$



$$G_1(s) = \frac{1 + Ts}{1 + T_1 s}$$

$$G_2(j\omega) = \frac{1 - j\omega T}{1 + j\omega T_1} \quad 0 < T < T_1$$



$$G_2(s) = \frac{1 - Ts}{1 + T_1 s}$$

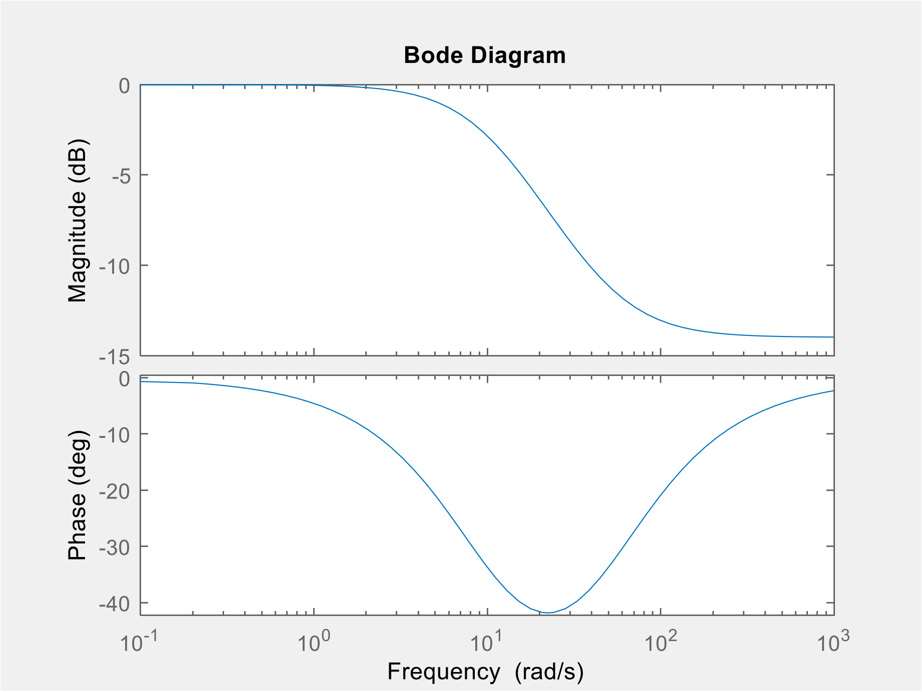
Frequency Response Analysis

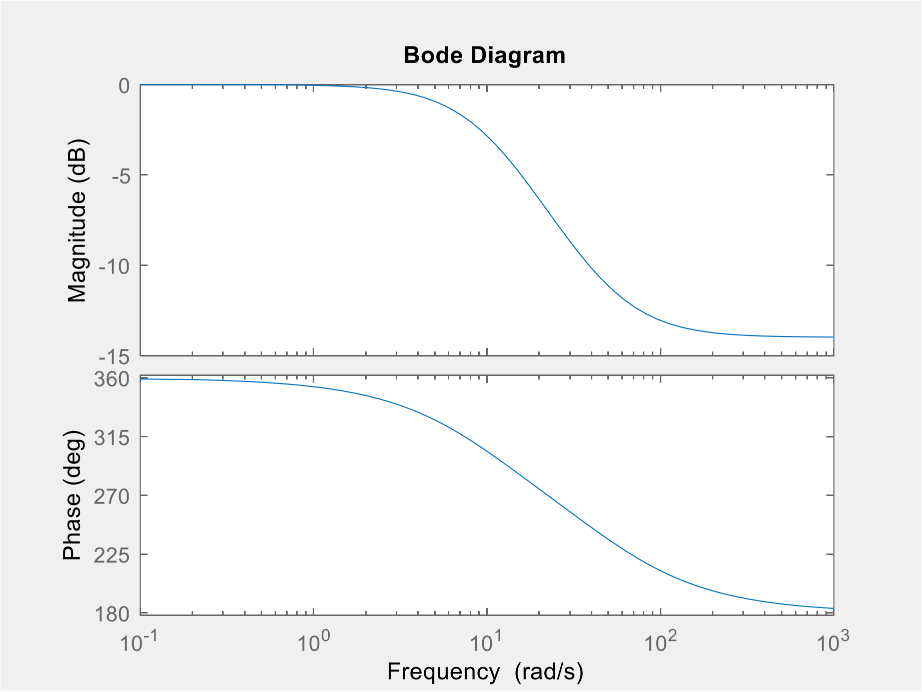
Bode Plot

Minimum-Phase Systems and Nonminimum-Phase Systems

$$G_1(j\omega) = \frac{1 + j\omega T}{1 + j\omega T_1}$$

$$G_2(j\omega) = \frac{1 - j\omega T}{1 + j\omega T_1} \quad 0 < T < T_1$$





Frequency Response Analysis

Bode Plot

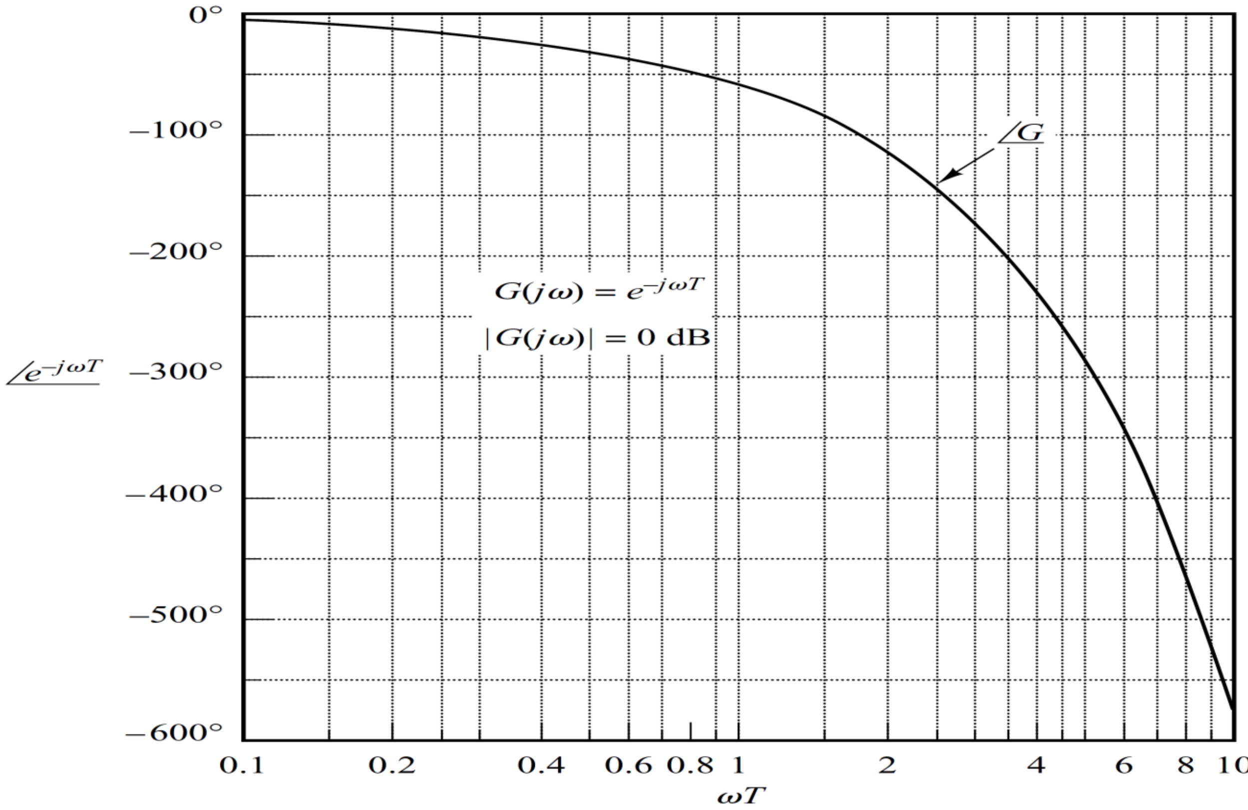
Transport Lag

Transport lag, which is also called **dead time**, is of **nonminimum-phase** behavior and has an excessive phase lag with no attenuation at high frequencies. Such transport lags normally exist in thermal, hydraulic, and pneumatic systems.

$$G(j\omega) = e^{-j\omega T}$$

$$|G(j\omega)| = |\cos \omega T - j \sin \omega T| = 1$$

$$\underline{/G(j\omega)} = -\omega T \qquad \text{(radians)}$$



Frequency Response Analysis

Bode Plot

Transport Lag

Example

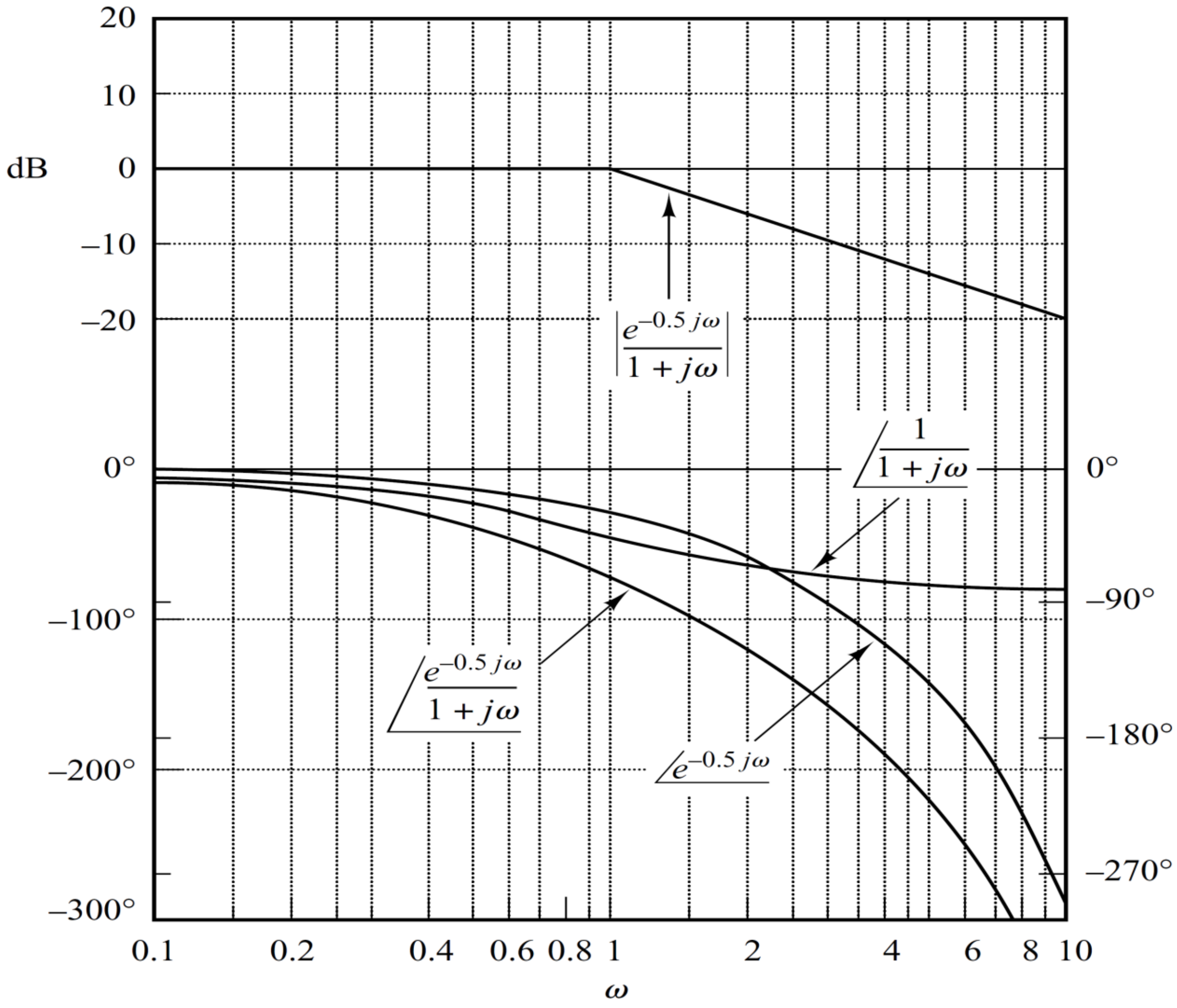
$$G(j\omega) = \frac{e^{-j\omega L}}{1 + j\omega T}$$

$$20\log|G(j\omega)| = 20\log\left|e^{-j\omega L}\right| + 20\log\left|\frac{1}{1 + j\omega T}\right|$$

$$= 0 + 20\log\left|\frac{1}{1 + j\omega T}\right|$$

$$\angle G(j\omega) = \angle e^{-j\omega L} + \angle \frac{1}{1 + j\omega T}$$

$$= -\omega L - \tan^{-1}\omega T$$



Frequency Response Analysis

Bode Plot

Resonant Frequency ω_r and the Resonant Peak Value M_r

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}, \qquad \text{for } 0 \le \zeta \le 0.707$$

- As the damping ratio ζ approaches zero, the resonant frequency approaches $\boldsymbol{\omega_n}$
- for ζ >0.707, there is no resonant peak.

$$\text{For } 0 \le \zeta \le 0.707 \longrightarrow M_r = |G(j\omega)|_{\max} = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

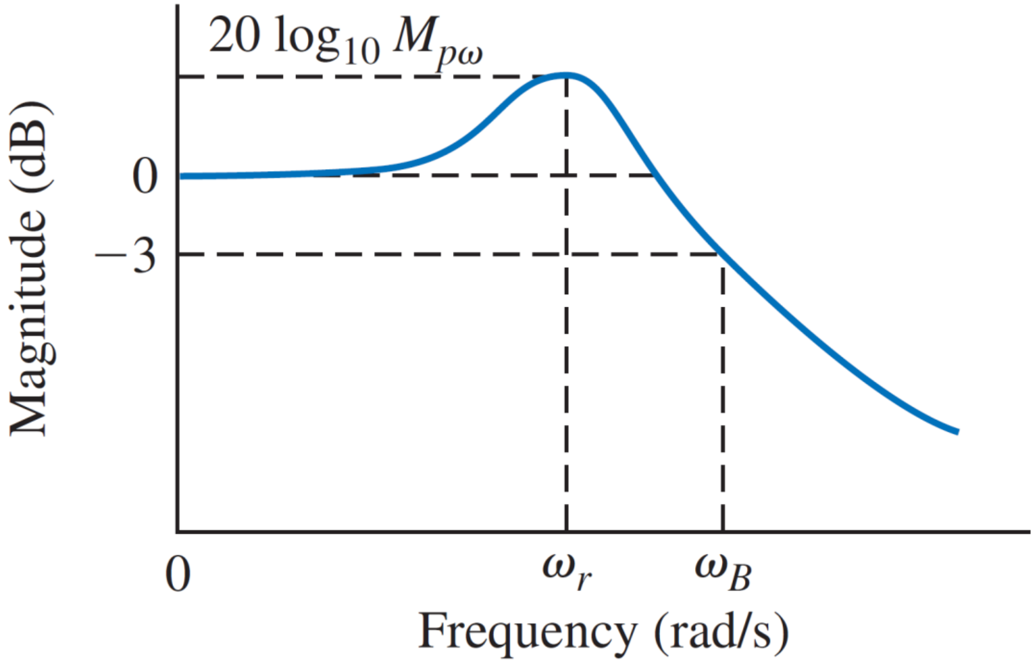
$$\text{For } \zeta > 0.707 \longrightarrow M_r = 1$$

- As ζ approaches zero, M_r approaches infinity. This means that if the undamped system is excited at its natural frequency, the magnitude of G(jω) becomes infinity.

Bode Plot

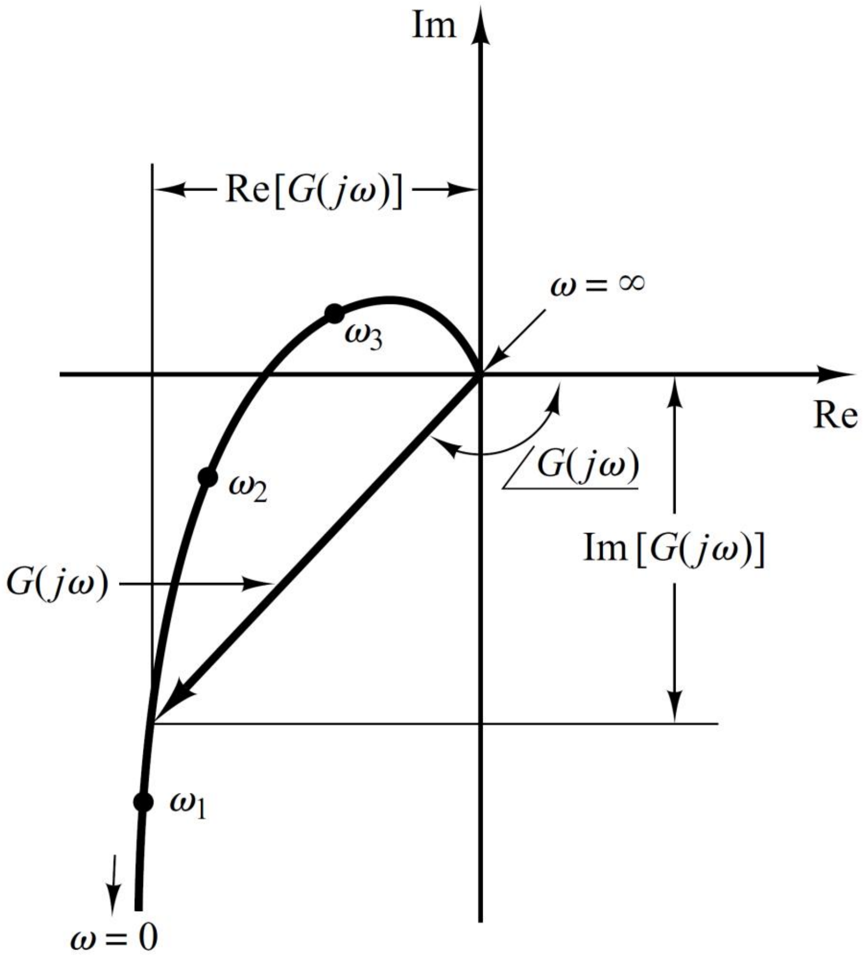
Resonant Frequency ω_r and the Resonant Peak Value M_r

- The bandwidth is the frequency ω_B at which the frequency response has declined 3 dB from its low-frequency value.
- This corresponds to approximately $\frac{1}{\sqrt{2}}$ of the low-frequency value.



Polar Plot

- A plot of the **magnitude** of ***G(jω)*** versus the **phase** angle of ***G(jω)*** on polar coordinates as ***ω*** is varied from zero to infinity.
- It is called **Nyquist** Plot if ***ω*** is varied from **negative** infinity to **positive** infinity.
- The Nyquist criteria help us determine the closed-loop system's **stability**.



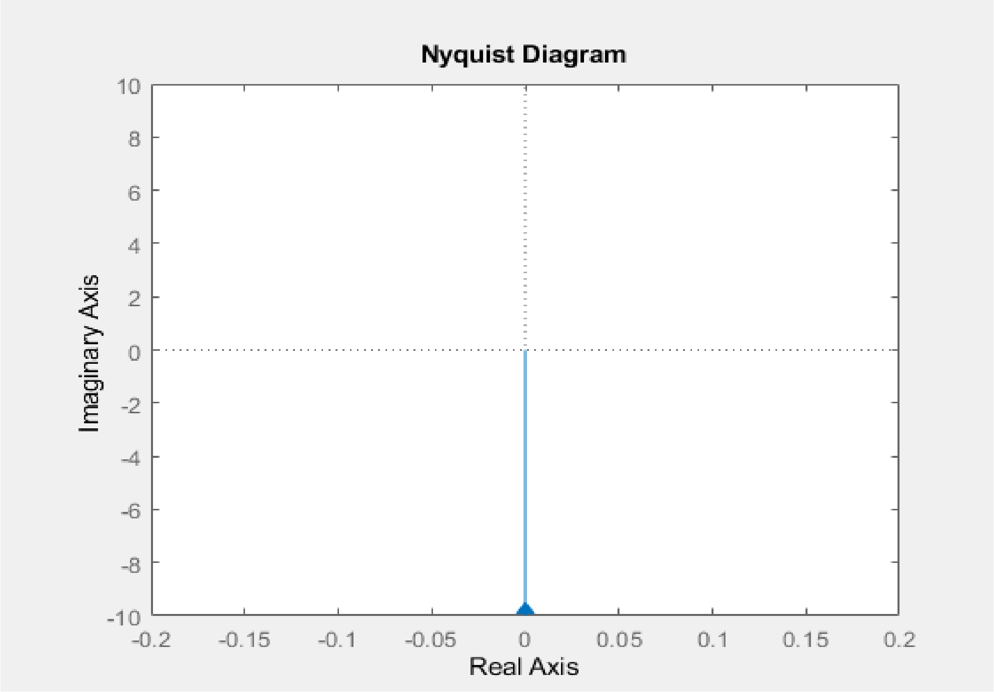
Frequency Response Analysis

Polar Plot

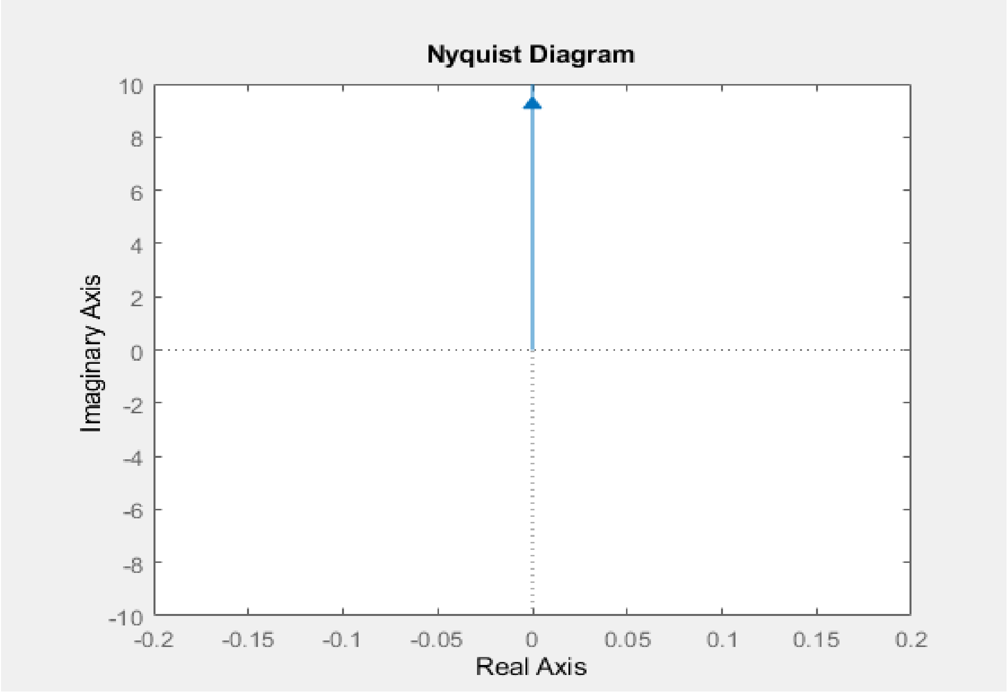
Integral and Derivative Factors $(j\omega)^{\pm}$

$$G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} = \frac{1}{\omega} \angle -90^\circ$$

The polar plot of G(jω)=1/jω is the negative imaginary axis



The polar plot of G(jω)=jω is the positive imaginary axis.



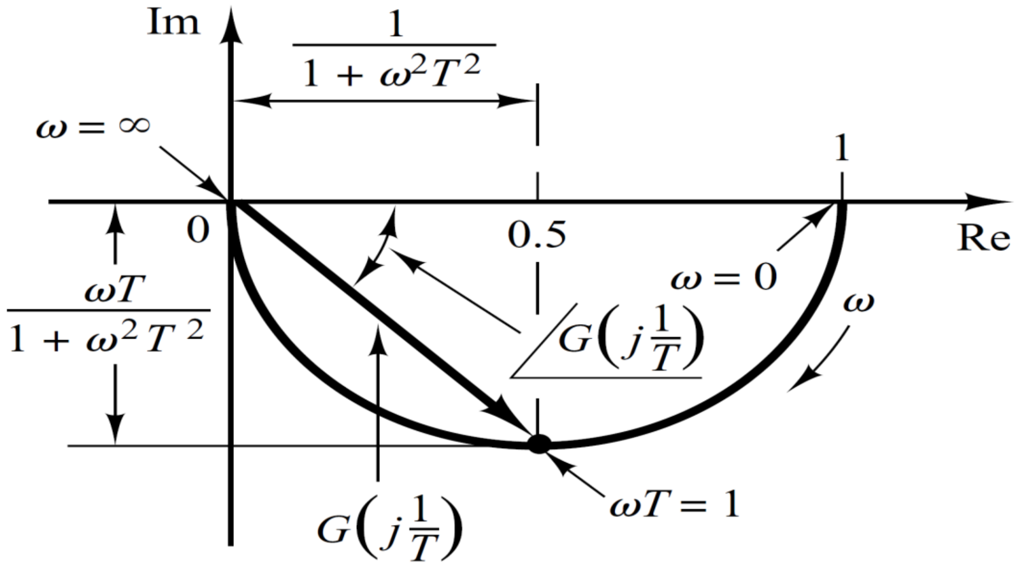
Frequency Response Analysis

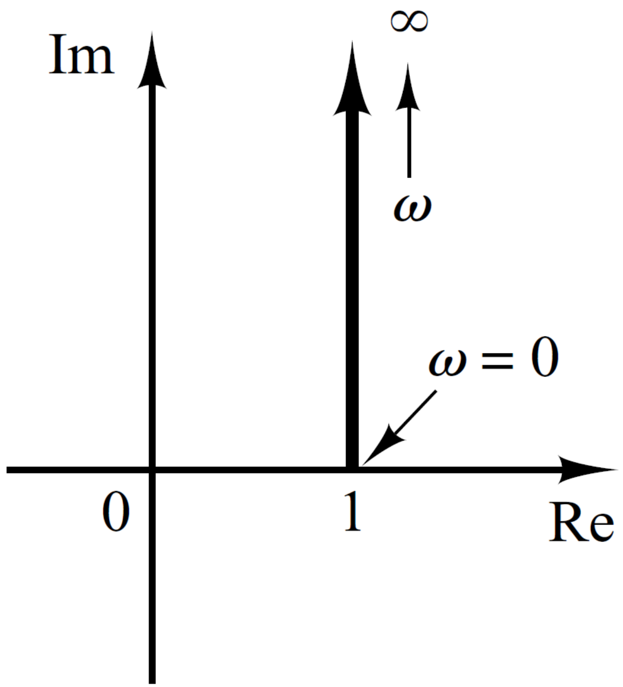
Polar Plot

First-Order Factors $(1 + j\omega)^{\pm}$

$$G(j\omega) = \frac{1}{1 + j\omega T} = \frac{1}{\sqrt{1 + \omega^2 T^2}} \angle -\tan^{-1} \omega T$$

$$G(j0) = 1 \angle 0^\circ \quad \text{and} \quad G\left(j\frac{1}{T}\right) = \frac{1}{\sqrt{2}} \angle -45^\circ$$





Polar plot of $1 + j\omega T$.

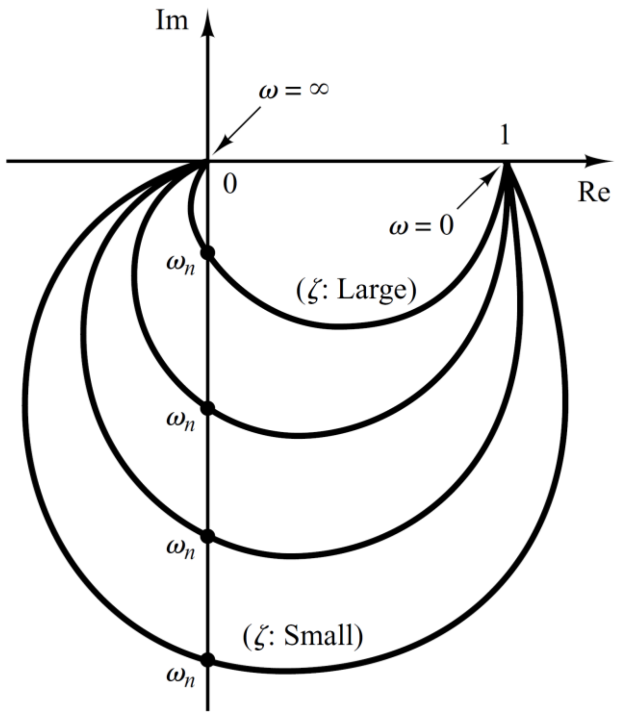
Frequency Response Analysis

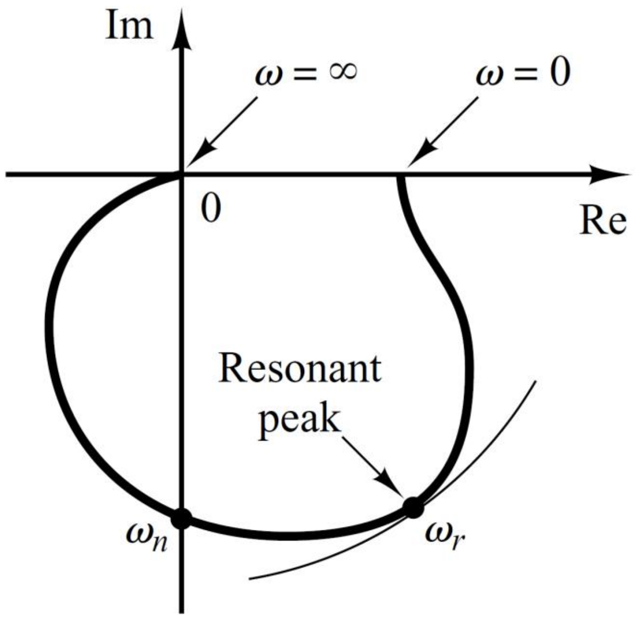
Polar Plot

Quadratic Factors $[1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2]^{\pm}$

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} \rightarrow \lim_{\omega\to 0} G(j\omega) = 1\angle 0^\circ \text{ and } \lim_{\omega\to\infty} G(j\omega) = 0\angle -180^\circ \leftarrow \text{for } \zeta > 0$$

$$\omega = \omega_n \text{ is } -90^\circ$$





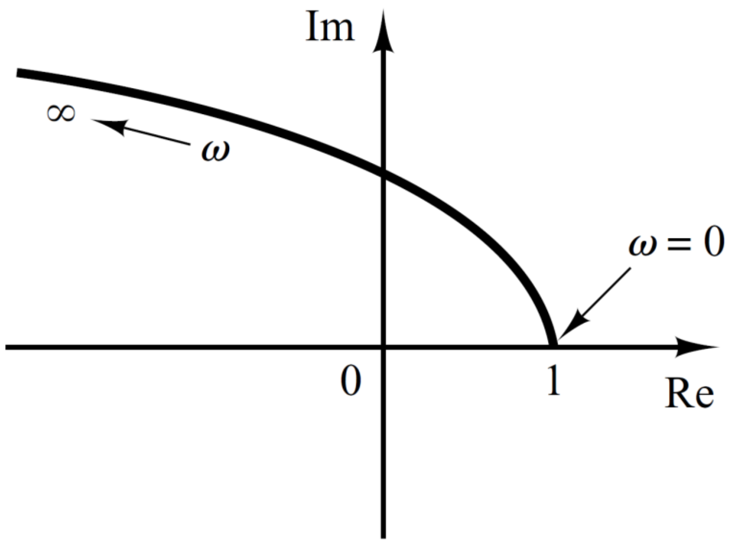
Frequency Response Analysis

Polar Plot

Quadratic Factors $[1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2]^{\pm}$

$$G(j\omega) = 1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2 = \left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\zeta\omega}{\omega_n}\right)$$

$$\lim_{\omega\to 0} G(j\omega) = 1\,\angle 0^\circ \qquad \lim_{\omega\to\infty} G(j\omega) = \infty\,\angle 180^\circ$$



Frequency Response Analysis

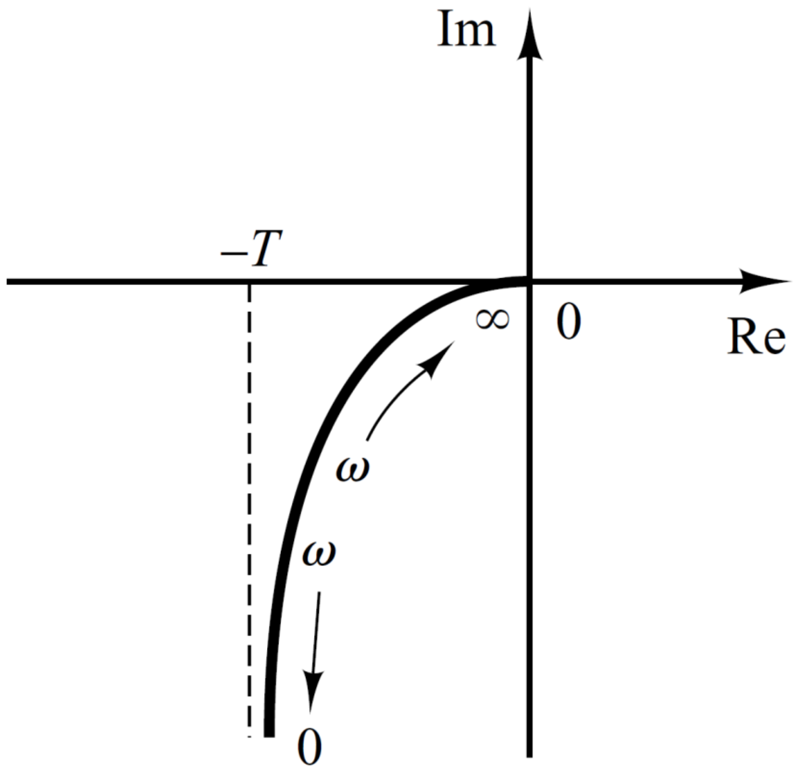
Polar Plot

Example

$$G(s) = \frac{1}{s(Ts+1)} \longrightarrow G(j\omega) = \frac{1}{j\omega(1+j\omega T)} = -\frac{T}{1+\omega^2 T^2} - j\frac{1}{\omega(1+\omega^2 T^2)}$$

$$\lim_{\omega \to 0} G(j\omega) = -T - j\infty$$

$$\lim_{\omega \to \infty} G(j\omega) = 0 - j0$$



Frequency Response Analysis

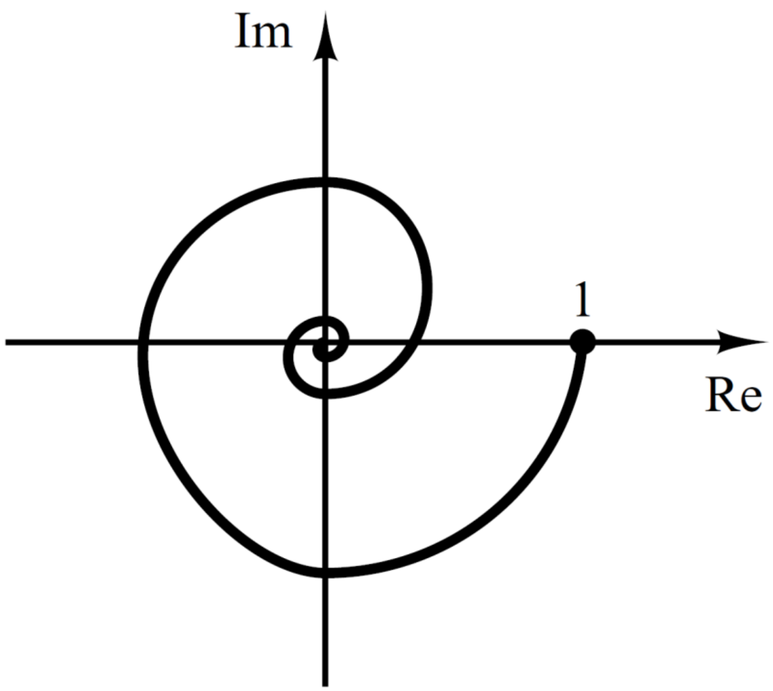
Polar Plot

Example

$$G(j\omega) = \frac{e^{-j\omega L}}{1 + j\omega T} \longrightarrow G(j\omega) = \left(e^{-j\omega L}\right)\left(\frac{1}{1 + j\omega T}\right)$$

$$|G(j\omega)| = |e^{-j\omega L}| \cdot \left|\frac{1}{1 + j\omega T}\right| = \frac{1}{\sqrt{1 + \omega^2 T^2}}$$

$$\underline{/G(j\omega)} = \underline{/e^{-j\omega L}} + \underline{/\frac{1}{1 + j\omega T}} = -\omega L - \tan^{-1} \omega T$$



Thank You

Any Questions ??